On higher regulators on Picard Modular Surface

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Outline

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2. Vanish on the boundary

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3.1. Statement of the result

3.2. Outline of the proof

1. Set up

- $E/\mathbb{Q}$: imaginary quad
  s.t. $\mathrm{disc}(E) = -D$

- $\mathcal{O}$: ring of integers in E

- $E \otimes_{\mathbb{Q}} \mathbb{R} \overset{\text{fix}}{\cong} \mathbb{C}$ s.t. $\delta = \sqrt{-D} > 0$

- $J = \begin{pmatrix} & \delta^{-1} \\ -\delta^{-1} & \end{pmatrix}$

- $G = GU(J)$

- $H = \{ (g, z) \in GL_2 \times \mathcal{O}^\times \mid \det g = z\bar{z} \}$

- $$\begin{tikzcd} H \arrow[r, hook, "\iota"] \arrow[d, "p"] & G \\ GL_2 & \end{tikzcd}$$

- V : alg repn of G

  W : alg repn of H

  s.t. $W \hookrightarrow \iota^* V$

$$
\begin{tikzcd}
H \arrow[r, hook, "\iota"] \arrow[d, "p"] & G \\
GL_2 &
\end{tikzcd}
$$

$$
\rightsquigarrow \quad
\begin{tikzcd}
M \arrow[d, phantom, "\shortparallel"] & S \arrow[d, phantom, "\shortparallel"] \\
Sh_H \arrow[r, hook, "\iota"] \arrow[d, "p"] & Sh_G \\
Sh_{GL_2} &
\end{tikzcd}
$$

quasi-projective$/E$

V : alg repn of G $\boxed{\text{Suff regular}}$

$$W = \mathrm{Sym}^n \, \mathrm{std}$$

s.t. $\boxed{W \hookrightarrow \iota^* V}$

"homological convention"

Notation

$$V \in VHS(S) \hookrightarrow D^b(MHM_{\mathbb{R}}(S))$$

$$V \longmapsto V[2]$$

[Ancona] [F. Jin]

$$V \in CHM(S) \hookrightarrow DM_{B,C}(S)$$

$$V \longmapsto V[0]$$

Similar for W, $\underline{\mathrm{Sym}^n \mathrm{std}}$ on GL_2

$\dim S = 2$ $\boxed{\text{Normalization}}$

$\boxed{\text{Def}}$

(1) $H^3_M(S, V(2)) := \mathrm{Hom}_{DM_{B,C}(S)}(1_S, V(2)[3])$

$H^1_M(M, W(1)) := \mathrm{Hom}_{DM_{B,C}(M)}(1_M, W(1)[1])$

(2) $H^3_H(S, V(2)) := \mathrm{Hom}_{D^b(MHM_{\mathbb{R}}(S))}(1_S, V(2)[3])$

$$H^1_{\mathcal{H}}(\mathcal{M}, W(1)) := \mathrm{Hom}_{D^b(MHM_{\mathbb{R}}(\mathcal{M}))}(\mathbb{1}_{\mathcal{M}}, W(1)[1])$$

$\boxed{\text{Prop}}$

(1) (Hodge realization) [J. Bouali] 22

$X = S, M$

$$R_H : DM_{B,c}(X) \longrightarrow D^b(MHM_{\mathbb{R}}(X))$$

$\rightsquigarrow$ Beilinson higher regulator

$$\boxed{\begin{aligned} r_H : \; & H^3_{\mathcal{M}}(S, V(2)) \longrightarrow H^3_{\mathcal{H}}(S, V(2)) \\ & H^1_{\mathcal{M}}(\mathcal{M}, W(1)) \longrightarrow H^1_{\mathcal{H}}(\mathcal{M}, W(1)) \end{aligned}}$$

(2) (Gysin morphism)

$$\boxed{\iota_* : H^1_{\mathcal{M}}(\mathcal{M}, W(1)) \longrightarrow H^3_{\mathcal{M}}(S, V(2))}$$

Pf : $W \hookrightarrow \iota^* V \in DM_{B,c}(M)$

$\quad\quad\quad\quad\cong$ absolute purity

$\quad\quad\quad\iota^! V(1)[2]$

Adjunction $\rightsquigarrow$ $\iota_! W \longrightarrow V(1)[2]$
$\Vert$ ι proper
$\iota_* W$

Apply $\rightsquigarrow$ $\mathrm{Hom}_{DM_{B,c}(S)}(\mathbb{1}_S, -(1)[1])$

$$\mathrm{Hom}_{DM_{B,c}(S)}(\mathbb{1}_S, \iota_* W(1)[1]) \longrightarrow \mathrm{Hom}_{DM_{B,c}(S)}(\mathbb{1}_S, V(2)[3])$$

$\Vert$ Adjunction

$\mathrm{Hom}_{DM_{B,c}(S)} = H^3_M(S, V(2))$

$$\mathrm{Hom}_{DM_{B,c}(M)}(\mathbb{1}_M, W(1)[1]) = H^1_M(M, W(1))$$

$\square$

Rmk: Similar for absolute Hodge Cohomology

(3) (pullback)

Modular Curve

$$\boxed{p^*: \quad H^1_M(Sh_{GL_2}, \mathrm{Sym}^n \mathrm{std}(1)) \longrightarrow H^1_M(M, W(1))}$$

---

2. Vanish on the Boundary

Construction of Motivic classes [Loeffler - Skinner - Zerbes 22]

$\mathbb{Q}$-v.s of ftns on $GL_2(\mathbb{A}_f)$

$$\begin{tikzcd} B_n \arrow[r, "\mathrm{Eis}^n_M"'] \arrow[rrr, bend right, "\mathcal{E}is^n_M"'] & H^1_M(Sh_{GL_2}, \mathrm{Sym}^n \mathrm{std}(1)) \arrow[r, "p^*"] & H^1_M(M, W(1)) \arrow[r, "\iota^*"] & H^3_M(S, V(2)) \end{tikzcd}$$

$$\boxed{C = \mathcal{E}is^n_M(\phi_f)} \quad \text{for some } \phi_f \in B_n$$

Notation : $\boxed{\mathcal{E}is^n_H} := R_H(\mathcal{E}is^n_M)$

2.1 Hodge Version

$$H^2_{B,!}(S, V(2)) := \mathrm{Im}(H^2_{B,c}(S, V(2)) \longrightarrow H^2_B(S, V(2)))$$

Thm (S. 2024) For $V = V^{a,b}\{r,s\}$ satisfying

(1) $\boxed{0 \leq -r \leq a \text{ and } 0 \leq -s \leq b}$ ("$GL_2 \subset GL_3$ Branching")

(2) $\boxed{a > 0 \text{ and } b > 0}$ "regular"

(3) $r \neq 0$ or $s \neq 0$

the map $\mathcal{E}is^n_H : B_{n,\mathbb{R}} \longrightarrow H^3_H(S, V(2))$ factors through

related to "cohomological" cusp auto repn

$$\mathrm{Ext}^1_{MHS^+_{\mathbb{R}}}(1, H^2_{B,!}(S, V(2))) \hookrightarrow H^3_H(S, V(2))$$

where
- $MHS^+_{\mathbb{R}}$: Abelian category of mixed $\mathbb{R}$-Hodge structures
- 1: unit in $MHS^+_{\mathbb{R}}$

2.2 Motivic Version

Thm (S. 2024) under the similar condition of previous thm,

the map $\mathcal{E}is^n_{\mathcal{M}} : B_n \longrightarrow H^3_{\mathcal{M}}(S, V(2))$

factors through

$$H^{3+a-b+3(r-s)}_{\mathcal{M}}(Gr_0 M_{gm}(S, V), \mathbb{Q}(2+a+2r-s))$$

$$\hookrightarrow H^3_{\mathcal{M}}(S, V(2))$$

Rmk First Result about motivic Version of Vanishing on the boundary

2.3 Outline of the pf

$$\begin{tikzcd}
S \arrow[r, hook, "j"] & S^* & \partial S \arrow[l, hook', "i"'] \\
M \arrow[u, "\iota"] \arrow[r, hook, "j'"] & M^* \arrow[u, "p"] & \partial M \arrow[l, hook', "i'"'] \arrow[u, "q"]
\end{tikzcd}$$

Baily-Borel compactification

a-dim

a-dim

Pf of Hodge Version

Step I

$\boxed{\text{Prop}}$. We have exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^2_{B,!}(S, V(2)) \longrightarrow H^3_{\mathcal{H}}(S, V(2)) \longrightarrow H^1_{H}(\partial S, i^* j_* V(2)).$$

Pf: formal operation of mixed Hodge module.

Step II

$\boxed{\text{Prop}}$

$$\begin{tikzcd} B_{n,\mathbb{R}} \arrow[d] & \\ H^1_H(M, W(1)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & H^0_H(\partial M, i'^* j'_* W(1)) \arrow[d, "\boxed{\theta_H}"] \\ H^3_H(S, V(2)) \arrow[r] & H^1_H(\partial S, i^* j_* V(2)) \end{tikzcd}$$

Pf formal operation of Mixed Hodge modules.

Step III

$\boxed{\text{Prop}}$ $\boxed{\theta_H = 0}$

Pf: from step I, suffice to show

$$\left(g_* i'^* j'_* W[-1] \longrightarrow i^* j_* V(1) \right) \in D^b(MHM^+_{\mathbb{R}}(\partial S))$$

$$\| \\ 0$$

$g_! = g_*$ $\quad$ $g^! = g^*$

$$\mathrm{Hom}_{D^b(MHM^+_{\mathbb{R}}(\partial S))}\left(\underset{\underset{g_!}{\|}}{g_*} i'^* j'_* W[-1],\ i^* j_* V(1) \right)$$

$$= \mathrm{Hom}_{D^b(MHM^+_{\mathbb{R}}(\partial M))}\left(i'^* j'_* W[-1],\ \underset{}{\overset{\overset{g^*}{\|}}{g^!}} i^* j_* V(1) \right)$$

[Burgos-Wildeshaus 04]

$$i'^* j'_* W = \bigoplus_n \mathcal{H}^n i^{*\prime} j'_* W[-n]$$

$$i^* j_* V = \bigoplus_n \mathcal{H}^n i^* j_* V[-n]$$

Cohomology of nilpotent Lie alg

$\Downarrow$

Kostant thm

Computation of degeneration of VHS

(Anona)

explicit computation using $\boxed{\text{Pink thesis}}$

$\square$

Motivic Version

$$V = V^{a,b}\{r,s\}$$

$\chi_1^a \chi_2^b \chi_3^r \chi_4^s$

$$\chi_1 : \begin{pmatrix} x & & \\ & z & \\ & & \frac{z\bar{z}}{\bar{x}} \end{pmatrix} \longmapsto x$$

$$\chi_2 : \begin{pmatrix} x & & \\ & z & \\ & & \frac{z\bar{z}}{\bar{x}} \end{pmatrix} \longmapsto \bar{x}$$

$$\chi_3 : \begin{pmatrix} x & & \\ & z & \\ & & \frac{z\bar{z}}{\bar{x}} \end{pmatrix} \longmapsto \frac{\det}{\nu}$$

$$\chi_4 : \begin{pmatrix} x & & \\ & z & \\ & & \frac{z\bar{z}}{\bar{x}} \end{pmatrix} \longmapsto \overline{\frac{\det}{\nu}}$$

$$G_E \cong GL_{3,E} \times G_{m,E}$$

$$V \longmapsto (a+r-s,\ r-s,\ -b+r-s\ ;\ b+2s-r)$$

$\boxed{\text{interior motive}}$ $\boxed{Gr_0 M_{gm}(V)}$

[J. wildeshaus] 15

$k = \min\{a,b\}$ $\quad$ $a>0$ and $b>0$

$\boxed{\text{FACT}}$ have dist Δ in $DM_{gm}(\operatorname{spec} E)$

$$C_{-(k+1)} \longrightarrow M_{gm}(V) \longrightarrow Gr_0 M_{gm}(V) \xrightarrow{+1} C_{-(k+1)}[1]$$

A/S universal Abelian 3-fold

$\boxed{\text{Prop}}$: (1) $H^3_M(S, V(2)) \subset H_M^{3+a-b+3(r-s)}(A^{a-b+3(r-s)}, \mathbb{Q}(a+2+2r-s))$

(2) $r_H(H_M^{3+a-b+3(r-s)}(Gr_0 M_{gm}(V), \mathbb{Q}(2+a+2r-s))$

$= \mathrm{Ext}^1_{MHS^+_{\mathbb{R}}}(1, H^2_{B,!}(S, V(2)))$

$\boxed{\text{Pf}}$: (1) $V = V(a+r-s, r-s, r-s-b; 2s-r+b)$

$\rightsquigarrow V \subset (\mathrm{std})^{\otimes(a-b+3(r-s))}(2s-r+b)$

Ancona
$\xrightarrow{\sim}$ $M(V/S) \subset h^{a-b+3(r-s)}(A^{a-b+3(r-s)}/S)(a+2r-s)$
Fangzhou Jin

$p: A^{a-b+3(r-s)} \longrightarrow S$

motivic
$\xrightarrow{\sim}$ $p_* \mathbb{Q}(a) \cong \bigoplus_i h^i(A^{a-b+3(r-s)})[-i]$
decomposition
thm

$\rightsquigarrow M(V/S) \subset h^{a-b+3(r-s)}(A^{a-b+3(r-s)}/S)(a+2r-s)$

$\subset (p_* \mathbb{Q}(a))[a-b+3(r-s)](a+2r-s)$

$\pi: S \longrightarrow \mathrm{Spec}\, E$

$\rightsquigarrow H^3_M(S, V(2)) = \mathrm{Hom}_{DM_{Bc}(S)}(\pi^* \mathbb{1}, M(V/S)(2)[3])$

$$\subset \mathrm{Hom}_{DM_{gm}(\mathrm{Spec}\,E)}(\mathbb{1}, (\pi_* p_* \mathbb{Q}(0))(a+2+2r-s)[3+a-b+3(r-s)])$$

$$= H_M^{3+a-b+3(r-s)}(A^{a-b+3(r-s)}, \mathbb{Q}(a+2+2r-s))$$

$\boxed{(2)}$

$$H_M^{3+a-b+3(r-s)}(\mathrm{Gr}_0 M_{gm}(V), \mathbb{Q}(2+a+2r-s))$$

$$\|$$

$$\mathrm{Hom}_{DM_{gm}(\mathrm{Spec}\,E)}(\mathrm{Gr}_0 M_{gm}(V), \mathbb{Q}(2+a+2r-s)[3+a-b+3(r-s)])$$

$$\Big\downarrow R_H \quad \text{Contravariant}$$

$$\mathrm{Hom}_{D^b(MHS_{\mathbb{R}}^+)}(\mathbb{R}(0), R_H(\mathrm{Gr}_0 M_{gm}(V))(2+a+2r-s)[3+a-b+3(r-s)])$$

$MHS_{\mathbb{R}}^+$ has cohomology dim 1

exact sequence

$$\rightsquigarrow 0 \longrightarrow \mathrm{Ext}^1_{MHS_{\mathbb{R}}^+}(1, H^2_{B,!}(S, V(2)))$$

$$\longrightarrow \mathrm{Hom}_{D^b(MHS_{\mathbb{R}}^+)}(\mathbb{R}(0), R_H(\mathrm{Gr}_0 M_{gm}(V))(2+a+2r-s)[3+a-b+3(r-s)]$$

$$\longrightarrow \mathrm{Hom}_{MHS_{\mathbb{R}}^+}(\mathbb{R}(0), \underbrace{H^3_{B,!}(S, V(2))}_{\substack{\| \\ 0 \\ \text{Saper thm}}}) \longrightarrow 0$$

$\square$

Step I

$$
\begin{tikzcd}
M \arrow[r, "j'"] \arrow[d, "\iota"] & M^* \arrow[d, "p"] & \partial M \arrow[l, "i'"'] \arrow[d, "q"] \\
S \arrow[r, "j"] & S^* & \partial S \arrow[l, "i"']
\end{tikzcd}
$$

$\boxed{\text{Prop}}$ We have the following exact sequence

$$
\boxed{
\begin{aligned}
0 &\longrightarrow H_{\mathcal{M}}^{3+a-b+3(r-s)}\left(Gr_0 M_{gm}(V), \mathbb{Q}(2+a+2r-s)\right) \\
&\longrightarrow H^3_{\mathcal{M}}(S, V(2)) \longrightarrow H^3_{\mathcal{M}}\left(\partial S, i^* j_* V(2)\right)
\end{aligned}
}
$$

$\boxed{\text{Pf}}$: exact $\triangle$ in $DM_{gm}(\operatorname{Spec} E)$

$$
C_{-(k+1)} \longrightarrow M_{gm}(V) \longrightarrow Gr_0 M_{gm}(V) \xrightarrow{+1} C_{-(k+1)}[1]
$$

$\rightsquigarrow$ Apply $\operatorname{Hom}_{DM_{gm}(\operatorname{Spec} E)}\left(-, \mathbb{Q}(a+2+2r-s)[3+a-b+3(r-s)]\right)$

$$
\begin{aligned}
\rightsquigarrow \quad & H_{\mathcal{M}}^{2+a-b+3(r-s)}\left(C_{\le -(k+1)}, \mathbb{Q}(2+a+2r-s)\right) \\
&\longrightarrow H_{\mathcal{M}}^{3+a-b+3(r-s)}\left(Gr_0 M_{gm}(V), \mathbb{Q}(2+a+2r-s)\right) \\
&\longrightarrow H_{\mathcal{M}}^{3+a-b+3(r-s)}\left(M_{gm}(V), \mathbb{Q}(2+a+2r-s)\right) = H^3_{\mathcal{M}}(S, V(2)) \\
&\longrightarrow H_{\mathcal{M}}^{3+a-b+3(r-s)}\left(C_{\le -(k+1)}, \mathbb{Q}(2+a+2r-s)\right)
\end{aligned}
$$

$$C_{\leq -(k+1)} \longrightarrow \partial M(U) \longrightarrow C_{\geq k} \xrightarrow{+1} C_{\leq -(k+1)}[1]$$

Apply $\rightsquigarrow$

$\mathrm{Hom}_{DM_{Sm}(\mathrm{Spec}\,E)}(-, \mathbb{Q}(a+2+2r-s)[3+a-b+3(r-s)])$

exact sequence $\rightsquigarrow$

$$\boxed{H_M^{2+a-b+3(r-s)}(C_{\geq k}, \mathbb{Q}(2+a+2r-s))} = 0$$

$$\longrightarrow H_M^{3+a-b+3(r-s)}(C_{\leq -(k+1)}, \mathbb{Q}(2+a+2r-s)) = H_M^3(\partial S, i^* j_* V(2))$$

injective

$$\longrightarrow H_M^{3+a-b+3(r-s)}(\partial M(U), \mathbb{Q}(2+a+2r-s))$$

Apply R_H

$$\boxed{H_H^{2+a-b+3(r-s)}(C_{\geq k}, \mathbb{Q}(2+a+2r-s))}$$

$$\| \\ 0$$

$\boxed{S : \partial S \to \mathrm{Spec}\, E}$

By $\boxed{\text{conservativity of chow motives of Abelian type}}$

$$H_M^{2+a-b+3(r-s)}(C_{\leq -(k+1)}, \mathbb{Q}(2+a+2r-s)) \Leftarrow \text{Similarly we have}$$

$$\| \\ 0$$

$\square$

Step II

$\boxed{\text{Prop}}$ We have

$$\begin{tikzcd}
B_n \arrow[d] & \\
H^1_M(M, W(1)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & H^1_M(\partial M, i'^* j'_* W(1)) \arrow[d, "\theta_M"] \\
H^3_M(S, V(2)) \arrow[r] & H^3_M(\partial S, i^* j_* V(2))
\end{tikzcd}$$

Pf $\quad \iota_* W \longrightarrow V(1)[2] \in DM_{B,c}(S)$

$$\begin{tikzcd}
M \arrow[r, "j'"] \arrow[d, "\iota"] & M^* \arrow[d, "p"] & \partial M \arrow[l, "i'"'] \arrow[d, "q"] \\
S \arrow[r, "j"] & S^* & \partial S \arrow[l, "i"']
\end{tikzcd}$$

$\overset{j_*}{\rightsquigarrow} \quad j_* \iota_* W \longrightarrow j_* V(1)[2] \in DM_{B,c}(S^*)$

$\overset{i_* i^*}{\rightsquigarrow} \quad i_* i^* \overset{p_* j'_*}{\underline{j_* \iota_*}} W \longrightarrow i_* i^* j_* V(1)[2] \in DM_{B,c}(S^*)$

p proper $\rightsquigarrow$ proper BC $\quad \underline{i^* p_*} = \underline{q_* i'^*}$

$\rightsquigarrow \quad i_* q_* i'^* j'_* W \longrightarrow i_* i^* \underline{j_* V(1)[2]} \in DM_{B,c}(S^*)$

Adjunction $\rightsquigarrow$ $\mathrm{id} \to i_* i^*$

$$\rightsquigarrow \quad \begin{array}{ccc} p_* j'_* W & \longrightarrow & i_* 2_* i'^* j'_* W \\ \downarrow & & \downarrow \\ j_* V(1)[2] & \longrightarrow & i_* i^* j_* V(1)[2] \end{array}$$

$\rightsquigarrow$ Apply $\mathrm{Hom}_{DM_{B,c}(S^*)}(1_{S^*}, -(1)[1])$

$$\rightsquigarrow \quad \begin{tikzcd} H^1_M(M, W(1)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & H^1_M(\partial M, i'^* j'_* W(1)) \arrow[d] \\ H^3_M(S, V(2)) \arrow[r] & H^3_M(\partial S, i^* j_* V(2)) \end{tikzcd}$$

$\square$

Step III

$\boxed{\text{Prop}}$ $\boxed{\theta_M = 0}$

<u>**Pf**</u>: $\left.\begin{array}{l} \theta_H = 0 \\ + \text{ Conservativity of Chow motives of Abelian type} \end{array}\right\} \Longrightarrow \theta_M = 0$

key: $\dim \partial S = 0$

$\square$

3. Connection to L-value

3.1 statement of the results

$$\mathrm{Eis}^n_M : B_n \longrightarrow H_M^{3+a-b+3(r-s)}(\underbrace{\mathrm{Gr}_0 \mathrm{Mgm}(V)}_{\text{interior motive}}, \mathbb{Q}(2+a+2r-s))$$

$$\hookrightarrow H^3_M(S, V(2))$$

(Pure) chow motive

$$\boxed{\mathrm{Gr}_0 \mathrm{Mgm}(V)} \xrightarrow{R_B} H^2_{B,!}(S, V)$$

under condition V reg

$$H^2_{B,!}(S,V) \cong H^2_{cusp}(S, V) = \bigoplus_{\pi = \pi_\infty \otimes \pi_f} m(\pi)\, \underline{H^2(\mathfrak{g}_\mathbb{C}, K_\mathbb{C}; V \otimes \pi_\infty)} \otimes \pi_f$$

$m(\pi) \leq 1$ — Arthur multiplicity formula

π_∞: DS

L-packet

$$P(V) := \{ \pi_\infty|_{G(\mathbb{R})^+} \ \mathrm{DS} \mid \text{s.t. } H^2(\mathfrak{g}_\mathbb{C}, K_\mathbb{C}; V \otimes \pi_\infty) \neq 0 \}$$

$$= \{ \pi_1, \pi_2, \pi_3, \overline{\pi_1}, \overline{\pi_2}, \overline{\pi_3} \}$$

π_2: $\boxed{\text{Generic}}$

Grothendieck motive associated to $\pi_f \otimes \pi_\infty$

$$\mathrm{Gr}_0 \mathrm{Mgm}(V) \xrightarrow{\text{Hecke op}} \boxed{M(\pi_f, V)}$$

$\boxed{\text{Rmk}}$ Hecke correspondence does NOT keep rat equivalence

Want to study Beilinson's Conjecture of $\boxed{M(\pi_f, V(2))}$

(Hodge decomposition)

$\boxed{\text{Prop}}$ $M_B(\pi_f, V(2))_{\mathbb{C}}$

$$\cong M_B^{-r,-2-a-b-s} \oplus M_B^{-1-a-r,-1-b-s} \oplus M_B^{-2-a-b-r,-s}$$
$$\oplus M_B^{-2-a-b-r,-r} \oplus M_B^{-1-b-s,-1-a-r} \oplus M_B^{-s,-2-a-b-r}$$

pf: $\boxed{\text{explicit computation}}$

$wt = -2-a-b-r-s \leq -3$

$\boxed{\text{Rmk}}$

$\boxed{\text{FACT}}$ we have $\boxed{\text{exact sequence}}$

2-dim

3-dim

$H^2_{B,!}(S, V(2))[\pi_f]$

$\parallel$

$$0 \to \underline{F^0 M_{dR}(\pi_f, V(2))} \to \underline{M_B^-(\pi_f, V(2))(-1)}$$
$$\to \operatorname{Ext}^1_{MHS^+_{\mathbb{R}}}(1, \underline{M_B(\pi_f, V(2))}) \to 0$$

$\boxed{\text{1-dim}}$

$\boxed{\text{Beilinson higher regulator}}$

$E(\pi_f) \rightsquigarrow$ rational field of

$\pi = \pi_f \otimes \pi_\infty$

$$\begin{tikzcd}
H_M^{3+a-b+3(r-s)}(M(\pi_f, V), \mathbb{Q}(2+a+2r-s)) \arrow[r, "r_H"] \arrow[d, hook] & \mathrm{Ext}^1_{MHS^+_{\mathbb{R}}}(1, M_B(\pi_f, V)(2)) \arrow[d, hook] \\
H_M^{3+a-b+3(r-s)}(\mathrm{Gr}_0 M_{gm}(V), \mathbb{Q}(2+a+2r-s)) \arrow[r, "r_H"] \arrow[d, hook] & \mathrm{Ext}^1_{MHS^+_{\mathbb{R}}}(1, H^2_{B,!}(S, V(2))) \arrow[d, hook] \\
H^3_M(S, V(2)) \arrow[r, "r_H"] & H^3_H(S, V(2))
\end{tikzcd}$$

1-dim

over $\boxed{\mathrm{Ext}^1_{MHS^+_{\mathbb{R}}}(1, M_B(\pi_f, V)(2))}$

Beilinson $E(\pi_f)$-structure

$$\boxed{B(\pi_f, V(2))} = \det F^0_{E(\pi_f)} M_{dR}(\pi_f, V(2))^* \underset{E(\pi_f)}{\otimes} \det_{E(\pi_f)} M_B(\pi_f, V(2))^{-(-1)}_{\mathbb{R}}$$

$$\delta(\pi_f, V(2)) := \det(M_B(\pi_f, V(2))_{\mathbb{C}} \xrightarrow{\sim} M_{dR}(\pi_f, V(2))_{\mathbb{C}})$$

$$\boxed{D(\pi_f, V(2)) = (2\pi i)^3 \delta(\pi_f, V(2))^{-1} B(\pi_f, V(2))}$$

Deligne rational structure

1-dim $E(\pi_f)$-subspace

of rank 1 $E(\pi_f)\otimes\mathbb{R}$ submodule

$\mathrm{Ext}^1_{MHS^+_{\mathbb{R}}}(1, M_B(\pi_f, V)(2))$

$$B_n \xrightarrow{\mathcal{E}is^n_{\mathcal{M}}} H_{\mathcal{M}}^{3+a-b+3(r-s)}(\mathrm{Gr}_0 M_{gm}(V), \mathbb{Q}(2+a+2r-s))$$

$$\bigcup$$

Let $c = \mathcal{E}is^n_{\mathcal{M}}(\phi_f)$

depends on Vanish on the boundary $\Longrightarrow$ $\boxed{\mathrm{Span}_{\mathbb{Q}[G(\mathbb{A}_f)]}(c)}$

$\rightsquigarrow$ $\boxed{\mathrm{Span}_{\mathbb{Q}[G(\mathbb{A}_f)]}(c)[\pi_f]}$ $\boxed{\text{Subspace}}$

$$\subset H_{\mathcal{M}}^{3+a-b+3(r-s)}(M(\pi_f, V), \mathbb{Q}(2+a+2r-s))$$

$$\Big\downarrow r_H$$

$$\mathrm{Ext}^1_{MHS^+_{\mathbb{R}}}(1, M_B(\pi_f, V)(2))$$

$\boxed{K(\pi_f, V(2))}$ $\rightsquigarrow$ another $E(\pi_f)$-rational structure

what about relationship between $K(\pi_f, V(2))$ and $D(\pi_f, V(2))$?

Weak Beilinson's Conjectures

$\boxed{\text{Thm (S. 2024)}}$ under the Condition

(1) $0 \le -r \le a$ and $0 \le -s \le b$

(2) $a+r \ne b+s$

(3) $a > 0$ and $b > 0$

(4) $r \ne 0$ or $s \ne 0$

Then

$$\boxed{k(\Pi_f, V(2)) = c \cdot \mathcal{L}(M(\Pi_f, V)(2), 0)\, D(\Pi_f, V)(2)}$$

Here, $c \in (E(\Pi_f) \otimes \mathbb{C})^\times$

$\boxed{\text{Rmk}}$ • If $c \in E(\Pi_f)^\times$, get Weak Beilinson for $M(\Pi_f, V(2))$

or at least $\in \overline{\mathbb{Q}}^\times$

But have not proved

• Application $k = c \cdot \mathcal{L} \cdot D$

$$\left.\begin{array}{l} c \neq 0 \\ \mathcal{L}(M(\Pi_f, V)(2), 0) \neq 0 \quad \text{abs conv pt} \\ D \neq 0 \end{array}\right\} \begin{array}{l} \Longrightarrow k \neq 0 \\ \Longrightarrow \boxed{c \neq 0} \end{array}$$

A question in [LSZ 22]

• V = trivial no vanish on the boundary

Hence c cannot be used to prove

Weak Beilinson Conj for $M(\Pi_f, V(2))$

But possible to prove $\boxed{c \neq 0}$ for application in Euler System Construction

A. pollock + S. shah geometric part has an error

Can be fixed by recent work of [Burgos – Cauchi – Lemma – Rodrigues Jacinto 24]

3.2 Outline of the proof

$$0 \to \underbrace{F^0 M_{dR}(\pi_f, V(2))}_{2\text{-dim}} \to \underbrace{M_B^-(\pi_f, V(2))(-1)}_{3\text{-dim}} \to \underbrace{\mathrm{Ext}^1_{MHS^+_{\mathbb{R}}}(1, M_B(\pi_f, V(2)))}_{\boxed{1\text{-dim}}} \to 0$$

$H^2_{B,!}(S, V(2))[\pi_f] = M_B(\pi_f, V(2))$

$\widetilde{V_K} \in M_B^-(\pi_f, V(2))(-1) \longmapsto V_K \in K(\pi_f, V(2))$

$\widetilde{V_D} \in M_B^-(\pi_f, V(2))(-1) \longmapsto V_D \in D(\pi_f, V(2))$

How to prove?

$\boxed{\text{Construct}}$ $\varphi: M_B^-(\pi_f, V(2))(-1) \longrightarrow E(\pi_f) \otimes \mathbb{C}$

Linear

s.t. φ trivial on $F^0 M_{dR}(\pi_f, V(2))_{\mathbb{R}}$

Then $$\boxed{K(\pi_f, V(2)) = \frac{\varphi(\widetilde{V_K})}{\varphi(\widetilde{V_D})} D(\pi_f, V(2))}$$

$\boxed{\text{How to construct?}}$ "Poincaré duality"

$$H^2_{B,!}(S, V)(2) \otimes H^2_{B,!}(S, D[2,2])(-2) \to H^4_{B,!}(S, \mathbb{Q}[2,2]) \xrightarrow{tr} \mathbb{Q}(0)$$

$\rightsquigarrow \; M_B(\pi_f, V(2)) \otimes M_B(\widetilde{\pi_f}\,|\nu|^{-2}, \; D\{2,2\})\,(-2)$ $\longrightarrow E(\pi_f)(0)$

$G(\mathbb{A}_f)$ - equivariant

perfect

$$\begin{aligned} \mathbin{/\!/}\! S \quad & M_B^{a+b+r+2,\,s} \oplus \boxed{M_B^{a+r+1,\,b+s+1} \\ & \oplus M_B^{b+s+1,\,a+r+1}} \oplus M_B^{r,\,a+b+s+2} \\ & \oplus M_B^{s,\,a+b+r+2} \oplus M_B^{a+b+s+2,\,r} \end{aligned}$$

By Hodge type Construct $\omega \in M_B^{a+r+1,\,b+s+1}$

$\overline{\omega} \in M_B^{b+s+1,\,a+r+1}$

$$\Rightarrow \boxed{\Omega = \frac{1}{2}(\omega + \overline{\omega})} \in \left(M_B(\widetilde{\pi_f}\,|\nu|^{-2}, \; D\{2,2\})(-2)\right)^{-}(-1)$$

How to construct ω ?

Write explicit form in $\boxed{H^2(\mathfrak{g}_{\mathbb{C}}, K_G; \; D\{2,2\} \otimes \pi_2)}$

$\Rightarrow$ $\boxed{\omega, \overline{\omega} \text{ generic}}$

So we need to

① Compute $\langle r_H(c), \Omega \rangle$

② Compute $\langle \widetilde{\nu}_0, \Omega \rangle$

① Compute $\boxed{\langle r_H(c), \Omega \rangle}$

Suffice to compute $\boxed{\langle r_H(c), \omega \rangle}$ on one complex embedding

$$\begin{tikzcd} H^3_H(S, V(2)) \arrow[r, hook] \arrow[d] & H_H^{3+a-b+3(r-s)}(A^{a-b+3(r-s)}, \mathbb{R}(a+2+2r-s)) \arrow[d, "r_{H\to D}"] \\ H^3_D(S, V(2)) \arrow[r, hook, "\text{define}"] & H_D^{3+a-b+3(r-s)}(A^{a-b+3(r-s)}, \mathbb{R}(a+2+2r-s)) \end{tikzcd}$$

$:= \boxed{r_{H\to D}(H^3_H(S, V(2)))}$

Deligne - Beilinson Coh $\longrightarrow$

$\boxed{\text{Rmk}}$ No def of DB Coh with general coefficient

what is $\boxed{\text{DB - Cohomology ?}}$

$\boxed{\text{Def}}$ (1) $\overline{X}/\mathbb{C}$ proper smooth $\quad D = \overline{X} - X$ Normal crossing divisor

$j : X \hookrightarrow \overline{X}$

[Jannsen]

$H^n_D(X, \mathbb{R}(p))$ defined as the n-th hyper Coh of

DB Cohomology

$$\mathbb{R}(p)_D := \text{Cone}\left(Rj_* \mathbb{R}(p) \oplus F^p \Omega^*_{\overline{X}}(\log D) \longrightarrow Rj_* \Omega^*_X\right)[-1]$$

(2) [BCLR 24]

$$\mathbb{R}(p)_{\mathcal{D}} \simeq \mathrm{Cone}\left(F^p \mathcal{D}^*_{si} \longrightarrow \mathcal{D}^*_{si,\mathbb{R}(p-1)}\right)[-1]$$

So

$$\boxed{H^n_{\mathcal{D}}(X, \mathbb{R}(p)) \simeq \frac{\{(S,T) : dS = 0,\ dT = \pi_{p-1}(S)\}}{d(\tilde{S}, \tilde{T})}}$$

where $(S,T) \in F^p \mathcal{D}^n_{si}(\overline{X}) \oplus \mathcal{D}^{n-1}_{si,\mathbb{R}(p-1)}(\overline{X})$

and $\boxed{d(\tilde{S}, \tilde{T}) = (dS,\ dT - \pi_{p-1}(S))}$

Rmk (1) $\boxed{D^*_{si}}$: sheaves on $\overline{X}$ of $\boxed{\text{tempered currents}}$

$$D^{\mathrm{pre}}_{si} : U \longmapsto \underbrace{\Gamma_c(U, \mathcal{A}^{d-p, d-q}_{rd})^*}_{\text{rapid decreasing differential form}}$$

$U \subseteq \overline{X}$ open

(2) In our case $\underset{\overline{X}}{\tilde{\mathcal{A}}^{a-b+3(r-s)}}$ smooth projective toroidal compactification of $\mathcal{A}^{a-b+3(r-s)}$

(3) [BCLR24] $\iota : X' \hookrightarrow X$ codim c closed immersion

$$\overline{X'} = X' \cup D' \rightarrow \underline{\text{SNC divisor}}$$

extend to $\iota : \overline{X'} \hookrightarrow \overline{X}$

s.t. $\iota^{-1}(D) = D'$

GVariant

$\boxed{\text{Gysin map}}$: $H_D^n(X', \mathbb{R}(p)) \xrightarrow{\iota_*} H_D^{n+2c}(X, \mathbb{R}(p+c))$

$[(S,T)] \longmapsto [(\iota_* S, \iota_* T)]$

$(\iota_* T)(w) := T(\iota^* w)$

(4) $\boxed{\text{Prop}}$ $w \in A_{rd}^{2d-n}(\overline{X})$ smooth closed rD diff form of Hodge type $\{(a,b) \mid a, b > d-p\}$,

Then $(S,T) \longmapsto T(w)$ induces

$$\boxed{\langle -, w \rangle : H_D^{n+1}(X, \mathbb{R}(p)) \longrightarrow \mathbb{C}}$$

$\boxed{\text{Rmk}}$: key; Compared to Jansen's description

do not need to assume w extends to boundary

(Not clear if w generic)

$\boxed{\text{Application in our setting}}$

$$\begin{array}{ccc} & & B_{n,\mathbb{R}} \\ & & \big\downarrow \text{Eis}_H^n \\ \mathrm{Ext}^1_{MHS_{\mathbb{R}}^+}(1, M_B(\Pi_f, V(2))) & \longleftrightarrow & \mathrm{Ext}^1_{MHS_{\mathbb{R}}^+}(1, H^2_{B,!}(S, V(2))) \end{array}$$

$$\longleftrightarrow H^3_H(S, V(2)) \xrightarrow{r_{H\to D}} H^3_D(S, V(2))$$

$$\big\downarrow$$

$$H_D^{3+a-b+3(r-s)}(A^{a-b+3(r-s)}, \mathbb{R}(2+a+2r-s))$$

$r_D = r_{H\to D} \circ r_H$ The way to compute $\boxed{r_D(\mathrm{Eis}^n_M(\phi_f))}$

$$
\begin{tikzcd}
\phi_f \arrow[ddddd, mapsto] & B_n \arrow[r, "r_D"] \arrow[d, "\mathrm{Eis}^n_M"'] & B_{n,\mathbb{R}} \arrow[d, "\mathrm{Eis}^n_D"] \\
& H^1_M(Sh_{SL_2}, \mathrm{Sym}^n \mathrm{std}(1)) \arrow[r, "r_D"] \arrow[d, "p^*"'] & H^1_D(Sh_{SL_2}, \mathrm{Sym}^n \mathrm{std}(1)) \\
& \text{By Beilinson} & \boxed{(\mathrm{Eis}_B(\phi_f), \mathrm{Eis}_H(\phi_f))} \arrow[d, "p^*"] \\
& H^1_M(M, W(1)) \arrow[r, "r_D"] \arrow[d, "\iota^*"'] & H^1_D(M, W(1)) \\
& & (p^*\mathrm{Eis}_B(\phi_f), p^*\mathrm{Eis}_H(\phi_f)) \arrow[d, "\iota^*"] \\
& H^3_M(S, V(2)) \arrow[r, "r_D"] & H^3_D(S, V(2))
\end{tikzcd}
$$

view it as diff form

view it as tempered current

$c = \iota^* p^* \mathrm{Eis}^n_M(\phi_f) \longmapsto (\iota^* p^* \mathrm{Eis}_B(\phi_f), \iota^* p^* \mathrm{Eis}_H(\phi_f))$

Then we have

$$\underline{\underline{\langle r_D(c), w\rangle}} = \langle l^* p^* \mathrm{Eis}_H(\phi_f), w\rangle$$

$$= \underline{\langle p^* \mathrm{Eis}_H(\phi_f), l^* w\rangle}$$

$\tilde{\pi}$ $\quad l \rightsquigarrow w \quad \rightsquigarrow \mathrm{Eis}_H$

$$= \underbrace{I(\varphi, \Phi, \nu, s)}\Big|_{s = 1+a+b+r+s}$$

zeta integral of Gelbart PS ~~and~~ and PS

$\nu = (|\,|^{-n}\nu^0, \nu_2 = 1)$, $\nu_1 =$

finite order

$\boxed{\text{Def}}$ $\quad \varphi$ cusp form $\quad T \subset GL_2 \quad \nu$: Hecke character of T

$\varphi \in \pi$, $\quad T$ torus, $\quad \nu = (\nu_1, \nu_2)$

Jacquet Eisenstein Series

$$I(\varphi, \Phi, \nu, s) = \int_{H(\mathbb{Q}) Z_H(\mathbb{A}) \backslash H(\mathbb{A})} \varphi(g)\, \underline{E(g_1, \Phi, \nu, s)}\, dg$$

$E(g_1, \Phi, \nu, s) = P_1(g)$

$\boxed{\text{Prop}}$ (1) $\boxed{\text{unfolding}}$

$$I(\varphi, \Phi, \nu, s) = \prod_v I_v(W_v, \Phi_v, \nu_v, s)$$

$$I_v(W_v, \Phi_v, \nu, s) = \int_{U_2(\mathbb{Q}_v) \backslash H(\mathbb{Q}_v)} \nu_1(\det(g_{1,v}))\, \Phi_v((0,1) g_{1,v})\, (\nu_1 \nu_2)^{-1}(t)\, |t|_v^{2s}\, dg_v$$

whittaker function

② v nonarch, $v = p$ $\quad \pi_p$ unram $\quad p$ unram

$$\boxed{I_p(W_p, \Phi_p, \nu_p, s) = L_p(s, \Pi_p \times \nu_{1,p}, \mathrm{std})}$$

deg 6

③ ν non arch

$$\boxed{I_p(W_p, \Phi_p, \nu_p, 1+a+b+r+s) \in \overline{\mathbb{Q}}^\times}$$

④ ν arch can compute
$\| \infty$

$\boxed{I_\infty(W_\infty, \Phi_\infty, \nu_\infty, 1+a+b+r+s)}$ in terms product of Γ-ftn and classical whittaker ftn

$$\text{So } \langle r_D(c), \omega \rangle = \prod_{p<\infty} I_p(W_p, \Phi_p, \nu_p, 1+a+b+r+s)$$

explicit C

$$\times \boxed{I_\infty(W_\infty, \Phi_\infty, \nu_\infty, 1+a+b+r+s)}$$

$$= \prod_{p<\infty} I_p(W_p, \Phi_p, \nu_p', 1) \times \quad C$$

$\nu_p' = (\nu_1 = \nu^0, \nu_2' = 1)$

$$=_{\overline{\mathbb{Q}}^\times} L(1, \tilde{\pi}, \mathrm{std}) \quad C$$

not complete L-ftn

② Compute $\langle \tilde{\nu}_D , \Omega \rangle$

$\boxed{\text{Prop}}$ Assume that $\boxed{a+r \neq b+s}$

Deligne Period

We have $\langle \Omega , \tilde{\nu}_D \rangle_B = \overline{\mathbb{Q}}^\times c^-(\pi_f, V(2))^{-1}$

Pf: By def of $\boxed{\text{Deligne Period}}$ □

Combine $\langle \tilde{\nu}_K , \Omega \rangle$

and $\langle \tilde{\nu}_D , \Omega \rangle$

We get

$\boxed{\text{Thm}}$ under the Condition

(1) $0 \le -r \le a$ and $0 \le -s \le b$

(2) $a+r \neq b+s$ comes from $\langle \tilde{\nu}_D , \Omega \rangle$

(3) $a > 0$ and $b > 0$

(4) $r \neq 0$ or $s \neq 0$

We have

$\in (E(\pi_f) \otimes \mathbb{C})^\times$

$$k(\pi_f, V(2)) = c \cdot L(1, \tilde{\pi}, std) \quad D(\pi_f, V)(2)$$

Relate the L-function $\qquad n = a+b+r+s$

Prop (1) $L^S(s, \tilde{\pi}, std) = L^S(s+n, \pi, std)$

(2) $L(M(\pi_f, V)(2), s) = L(s+n+1, \pi, std)$

Pf (1) explicit MWV involution in [PS18]

(2) proved in [LR]

So $L(1, \tilde{\pi}, std) \longleftrightarrow L(1+n, \pi, std)$

$$\Updownarrow$$

$L(M(\pi_f, V)(2), 0)$