On higher regulators of Picard modular surfaces

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Euler's calculations

In the 1700s, Euler made the following famous computations:

- $$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6},$$
- $$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots=\frac{\pi^4}{90}.$$

Notice similar exponents.

Euler's calculations

Definition

Bernoulli numbers $B_k \in \mathbb{Q}$ are given by the expansion

$$\frac{t}{e^t - 1} = \sum_{k \geq 0} B_k \frac{t^k}{k!} = 1 - \frac{1}{2}t + \frac{1}{6} \cdot \frac{t^2}{2} - \frac{1}{30} \cdot \frac{t^4}{4!} + \cdots .$$

Euler showed the following formula:

$$1 + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \frac{1}{4^{2m}} + \cdots = \frac{(2\pi)^{2m} |B_{2m}|}{2(2m)!}, \quad \text{for } m \in \mathbb{Z}^+.$$

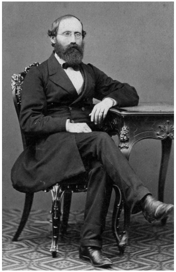
Examples

- $(m = 1)$ $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{(2\pi)^2}{2 \cdot 2} \cdot \frac{1}{6} = \frac{\pi^2}{6}$
- $(m = 2)$ $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{(2\pi)^4}{2 \cdot 4!} \cdot \frac{1}{30} = \frac{\pi^4}{90}$

Riemann ζ-function

In 1859, Riemann introduced the ζ-function of a complex variable: if $s \in \mathbb{C}$,

- $\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}$ for $\mathrm{Re}(s) > 1$.
- (Euler product):
  $\zeta(s) = \prod_p \frac{1}{1-\frac{1}{p^s}}$ for $\mathrm{Re}(s) > 1$.
- It has meromorphic continuation to $\mathbb{C}$.
- It has a (simple) pole only at $s = 1$.
- (Functional eqn): $\Lambda(s) = \Lambda(1-s)$
  for $\Lambda(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$.
  Call $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ a Γ-factor.



Dedekind ζ-function

Dedekind (1879) generalized $\zeta(s)$ to an arbitrary number field F.

- $\zeta_F(s) := \sum_{\mathcal{I}} \frac{1}{|\mathcal{O}_F/\mathcal{I}|^s}$, for $\mathrm{Re}(s) > 1$, where $\mathcal{I}$ runs over the non-zero ideals of O_F, so $\zeta_{\mathbb{Q}}(s) = \zeta(s)$.
- (Euler product): $\zeta_F(s) = \prod_{\wp} \frac{1}{1-\frac{1}{|\mathcal{O}_F/\wp|^s}}$ for $\mathrm{Re}(s) > 1$, where $\wp$ runs over the non-zero prime ideals of $\mathcal{O}_F$.
- $\zeta_F(s)$ has meromorphic continuation to $\mathbb{C}$.
- $\zeta_F(s)$ has a (simple) pole only at $s = 1$.
- (Functional eqn): $\Lambda_F(s) = \Lambda_F(1-s)$ for $\Lambda_F(s) :=$ $|d_F|^{\frac{s}{2}}(\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}))^{r_1}((2\pi)^{-s}\Gamma(s))^{r_2}\zeta_F(s)$.

Class number formula

The residue of $\zeta_F(s)$ at $s=1$ is related to global arithmetic invariants of F by the class number formula:

$$\mathrm{Res}_{s=1}\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}}{|d_F|^{\frac{1}{2}}\omega(F)}h(F)R(F) =_{\overline{\mathbb{Q}}^\times} (2\pi)^{r_2}R(F).$$

- d_F: discriminant of F
- $\omega(F)$: the number of roots of unity in F
- $h(F)$: class number of F
- $R(F)$: covolume of Dirichlet regulator map

$$r_{Dir} : O_F^\times \to \mathbb{R}^{r_1+r_2},$$

$\dim \mathrm{Im}(r_{Dir}) = r_1 + r_2 - 1.$
e.g. $F = \mathbb{Q}(\sqrt{2})$, $O_F^\times/\{\pm 1\} = (1+\sqrt{2})^{\mathbb{Z}}$,
$r_{Dir}(1+\sqrt{2}) = (\log(1+\sqrt{2})), -\log(1+\sqrt{2})),$
$R(F) = \log(1+\sqrt{2})$

BSD conjecture

- For an elliptic curve E over $\mathbb{Q}$, we can define its L-function $L(E,s)$ and regulator $R(E)$ similarly.
- Birch and Swinnerton–Dyer conjecture predicts that

$$\frac{L^{(r)}(E,1)}{r!} =_{\mathbb{Q}^\times} \Omega(E)R(E),$$

where

- $r = \mathrm{ord}_{s=1} L(E,s)$, so $\frac{L^{(r)}(E,s)}{r!}$ is lead. coeff. of $L(E,s)$ at $s=1$.
- $\Omega(E)$: the period of E

e.g. $E : y^2 = x^3 - 2$, $r = 1$, $E(\mathbb{Q})/E(\mathbb{Q})_{tor} = \langle P \rangle = \langle (3,5) \rangle$

$$\Omega(E) \approx 2.16368, R(E) = \widehat{h}(P) \approx 1.34957$$
$$\Omega(E)R(E) \approx 2.92003,\ L'(E,1) \approx 2.92005,$$
$$L'(E,1) = \Omega(E)R(E)$$

Beilinson's conjectures

In the 1980s, Beilinson made a deep conjecture about special values of motivic L-functions generalizing the classical analytic class number formula.

Let X be a smooth projective variety over $\mathbb{Q}$, $i \geq 0$ and $n \in \mathbb{Z}$ satisfying $2n > i$. Replace ingredients of class number formula:

- $O_F^\times \rightsquigarrow \mathrm{H}_M^{i+1}(X, \mathbb{Q}(n))$ (Motivic cohomology)
  - If $2n = i+1$, then $\mathrm{H}_M^{i+1}(X, \mathbb{Q}(n)) \cong \mathrm{CH}^n(X)_{\mathbb{Q}}$.
  - If $n = 1, i = 0$, then $\mathrm{H}_M^{i+1}(X, \mathbb{Q}(n)) = \mathrm{H}_M^1(X, \mathbb{Q}(1)) \cong \mathbb{Q}(X)^\times$.

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- $\mathbb{R}^{r_1+r_2} \rightsquigarrow \mathrm{H}_H^{i+1}(X, \mathbb{R}(n))$ (Absolute Hodge cohomology)

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- $\mathbb{R}^{r_1+r_2} \rightsquigarrow \mathrm{H}_H^{i+1}(X, \mathbb{R}(n))$ (Absolute Hodge cohomology)
- $r_{Dir} \rightsquigarrow r_H$

$r_H : \mathrm{H}_M^{i+1}(X, \mathbb{Q}(n)) \to \mathrm{H}_H^{i+1}(X, \mathbb{R}(n))$ (Beilinson's higher regulator)

Beilinson's conjectures

- $M = h^i(X)(n)$: a pure motive associated to X and n.
  $w = i - 2n$: its weight, so $2n > i$ implies $w < 0$.
- $\zeta_F(s) \rightsquigarrow L(M,s)$ (Motivic L-function),
  - For $\mathrm{Re}(s) > \frac{w}{2} + 1$, $L(M,s)$ is convergent Euler product.
  - A meromorphic cont. and functional equation of $L(M,s)$ relating s and $w+1-s$ is conjectured, mainly still open.
  - $w < 0 \Rightarrow w \leq -1$, so $0 \geq \frac{w+1}{2}$: center of $L(M,s)$.

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  - $w < 0 \Rightarrow w \leq -1$, so $0 \geq \frac{w+1}{2}$: center of $L(M, s)$.
- Critical points vs. Non-critical points
  Let $\Gamma_\infty(M, s)$ be associated Gamma factor of $L(M, s)$. Call $n \in \mathbb{Z}$ **critical** for $L(M, s)$ if it is not a pole of $\Gamma_\infty(M, s)$ or $\Gamma_\infty(M, w + 1 - s)$. Otherwise, $n \in \mathbb{Z}$ is called **non-critical**.

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  - For $\zeta(s)$, $w = 0$, $\Gamma_\infty(\mathbb{Q}, s) = \pi^{-s/2}\Gamma(s/2)$, critical points are positive even integers and negative odd integers. Non-critical points are positive odd integers and non-positive even integers.

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  - For $\zeta(s)$, $w = 0$, $\Gamma_\infty(\mathbb{Q}, s) = \pi^{-s/2}\Gamma(s/2)$, critical points are positive even integers and negative odd integers. Non-critical points are positive odd integers and non-positive even integers.
  - For $L(E, s)$, $w = 1$, $\Gamma_\infty(E, s) = 2(2\pi)^{-s}\Gamma(s)$, critical point is $s = 1$. Non-critical points are integers not equal to 1.

Beilinson's conjectures

- If $s = 0$ is *critical* for M, Deligne conjectured that

$$\boxed{L(M,0) \in c^+(M)\mathbb{Q}^\times},$$

where $c^+(M)$ is Deligne period.
e.g. If $M = \mathbb{Q}(2m)$ for $m \in \mathbb{Z}_{>0}$, then $L(M,s) = \zeta(s+2m)$ and $c^+(M) = (2\pi i)^{2m}$: Euler's $\zeta(2m)$-formula.

- If $s = 0$ is *non-critical* for M and $2n \geq i+3$, then $w = i - 2n \leq -3$ and $\frac{w}{2} + 1 \leq -\frac{1}{2} < 0$, so $L(M,0)$ makes sense as an Euler product. Beilinson conjectured that

$$\wedge^{\mathrm{top}} r_H(H^{i+1}_M(X, \mathbb{Q}(n))) =_{\mathbb{Q}^\times} L(M,0)\mathcal{D}(M),$$

where $\mathcal{D}(M)$ is the Deligne rational structure.
e.g. If $M = \mathbb{Q}(3)$, $L(M,s) = \zeta(s+3)$, so $L(M,0) = \zeta(3)$, $r_H = 2r_B$, where $r_B : K_5(\mathbb{Z}) \to \mathbb{R}$ is a Borel regulator.

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The setup

Notations

- Let E be an imaginary quadratic field of discriminant $-D$, and let $x \mapsto \bar{x}$ be the nontrivial Galois automorphism of E over $\mathbb{Q}$.
- Let $\mathcal{O}$ be the ring of integers of E.
- Fix an identification of $E \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}$ s.t. the imaginary part of $\delta := \sqrt{-D}$ is positive.

The group $G = GU(2,1)$

Definition

Let $J \in \mathrm{GL}_3(E)$ be the Hermitian matrix

$$J = \begin{pmatrix} 0 & 0 & \frac{1}{\delta} \\ 0 & 1 & 0 \\ -\frac{1}{\delta} & 0 & 0 \end{pmatrix}, \quad \text{where } \delta = \sqrt{-D},$$

and let $G = \mathrm{GU}(2,1)$ be the group scheme over $\mathbb{Z}$ such that for $\mathbb{Z}$-algebras R, we have for units $\mu \in R^\times$,

$$G(R) = \{(g, \mu) \in \mathrm{GL}_3(\mathcal{O} \otimes_{\mathbb{Z}} R) \times R^\times \,|\, {}^t\bar{g}Jg = \mu J\}.$$

Let H be the group scheme over $\mathbb{Z}$ such that for $\mathbb{Z}$-algebras R,

$$H(R) = \{(g, z) \in \mathrm{GL}_2(R) \times (\mathcal{O} \otimes_{\mathbb{Z}} R)^\times \,|\, \det(g) = z\bar{z}\}.$$

Modular curves

Definition

Let $\mathcal{H} = \{\tau = x + iy | x \in \mathbb{R}, y \in \mathbb{R}_{>0}\}$ be the upper half plane. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, acting on $\mathcal{H}$ by

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

The modular curve $Y(1)$ is defined as

$$Y(1) := \Gamma \backslash \mathcal{H}.$$

It is an affine algebraic curve over $\mathbb{Q}$.

Picard modular surfaces

- Picard modular surfaces are certain 2-dimensional Shimura varieties over E that generalize modular curves over $\mathbb{Q}$.
- $\mathcal{H} \rightsquigarrow$ complex 2-ball X in $\mathbb{C}^2$ ($|z_1|^2 + |z_2|^2 < 1$)
- $\mathrm{SL}_2(\mathbb{Z}) \rightsquigarrow \Gamma = \mathrm{GU}(2,1)(\mathbb{Z})$ (Picard modular group)
- Picard modular surface of level Γ is defined as $\mathrm{Sh}_G(\Gamma) := \Gamma \backslash X$
- Picard modular surfaces are algebraic surfaces over E. (Note E used to define J which appears in the definition of G.)

Charles Émile Picard

Goro Shimura

Galois representations

- For an elliptic curve $E/\mathbb{Q}$ which is defined by the equation $y^2 = x^3 + ax + b$, where $a, b \in \mathbb{Q}$, for a fixed prime p, its Tate module $T_p(E)$ is defined as

$$T_p(E) = \varprojlim_n E[p^n]$$

  where $E[p^n]$ is the p^n-torsion points of E.
- There is a natural action ρ_E of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $T_p(E)$ called the p-adic Galois representation associated to E.
- For a cusp form f with weight 2 and level $\Gamma_0(N)$, can define its Galois representation ρ_f.
- [C. Breuil-B. Conrad-F. Diamond-R. Taylor 1999] To each $E/\mathbb{Q}$, $\rho_E \cong \rho_f$ for some f of weight 2.
- Galois representations are étale realizations of motives.

Automorphic motives

- For a cusp form f, can construct its Grothendieck motive $M(f)$ by work of Scholl.
- $GL_2 \rightsquigarrow \mathrm{GU}(2,1)$
- $$f \rightsquigarrow \pi = \pi_f \otimes \pi_\infty,$$
  where π is some "cohomological" irreducible cuspidal automorphic representation of $GU(2,1)$.
- π can be thought as some kind of Picard modular form.
- $$M(f) \rightsquigarrow M(\pi_f, V),$$
  where the $M(\pi_f, V)$ is a Grothendieck motive associated to π.

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Outline

Beilinson's conjectures

When $2n \geq i+3$ and $\dim_{\mathbb{R}} H_H^{i+1}(X, \mathbb{R}(n)) = 1$,

$$\boxed{r_H(H_M^{i+1}(X, \mathbb{Q}(n))) =_{\mathbb{Q}^\times} L(M,0)\mathcal{D}(M)}.$$

Let $S := \mathrm{Sh}_G$, $G = \mathrm{GU}(2,1)$ and $M = \mathrm{Sh}_H$.

- **Step one**: Construct motivic classes c in $\mathrm{H}^3_M(S, V(2))$, where S is the Picard modular surface and V is some non-trivial nice "motivic local system" on it;
- **Step two**: Prove that the classes c lie in a "nice" subspace of $\mathrm{H}^3_M(S, V(2))$;
- **Step three**: Compute image of c under higher regulator r_H and relate to $L(M(\pi_f, V(2)), 0)$.

The L-value result I

Theorem (S. 2024)

For suitable non-trivial algebraic representations V of G, if we choose some "cohomological" irreducible cuspidal automorphic representation π of G that appears in $\mathrm{H}^2_{B,!}(S, V(2))$, we get:

$$\mathcal{K}(\pi_f, V(2)) = C \cdot L(M(\pi_f, V(2)), 0)\mathcal{D}(\pi_f, V(2))$$

where $C \in (E(\pi_f) \otimes_{\mathbb{Q}} \mathbb{C})^\times$,

- *$M(\pi_f, V(2))$ is a motive associated to π.*
- *$\mathcal{K}(\pi_f, V(2))$: 1-dim $E(\pi_f)$-subspace of a certain rank one $E(\pi_f) \otimes_{\mathbb{Q}} \mathbb{R}$-module generated by $r_H(c)$, c is the constructed motivic class in $\mathrm{H}^3_M(S, V(2))$.*
- *$\mathcal{D}(\pi_f, V(2))$: another 1-dim $E(\pi_f)$-subspace of the same $E(\pi_f) \otimes_{\mathbb{Q}} \mathbb{R}$-module, called the Deligne $E(\pi_f)$-structure.*

The L-value result II

Remark

- This result gives evidence towards Beilinson's conjectures.
- Constant C should be in $E(\pi_f)^\times$ but we have not proven it.
- $\mathcal{K}(\pi_f, V(2)) = C \cdot L(M(\pi_f, V(2)), 0)\mathcal{D}(\pi_f, V(2))$, $C \neq 0$, $L(M(\pi_f, V(2)), 0) \neq 0$ and $\mathcal{D}(\pi_f, V(2)) \neq \{0\}$, so we proved the motivic class c that generates the left side is non-trivial, which answers a question raised in [D. Loeffler-C. Skinner-S. Zerbes 2022]. In their paper, they *assume* the class c is non-trivial and use it to construct an Euler system for $\mathrm{GU}(2,1)$ based on the nontriviality.

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- If V is trivial, similar results were obtained in [A. Pollack-S. Shah 2018].
- Similar relations with non-trivial coefficients were obtained in [G. Kings 1998] and [F. Lemma 2017].

The construction of motivic classes

- **Starting point**: [Beilinson 83] The **Eisenstein symbol**:

$$B_n \xrightarrow{Eis^n_M} \mathrm{H}^1_M(\mathrm{Sh}_{\mathrm{GL}_2}, \mathrm{Sym}^n V_2(1)),$$

  where $\mathrm{Sh}_{\mathrm{GL}_2}$ is a modular curve. It can be seen as incarnation of real analytic Eisenstein series in the motivic world.
- Define the following two maps:

$$\iota : H \hookrightarrow G, \quad (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto (\begin{pmatrix} a & 0 & b \\ 0 & z & 0 \\ c & 0 & d \end{pmatrix}, z\bar{z})$$

  and

$$p : H \twoheadrightarrow \mathrm{GL}_2, \quad (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- The maps $\iota : H \hookrightarrow G$ and $p : H \twoheadrightarrow \mathrm{GL}_2$ of algebraic groups will induce the following morphisms of Shimura varieties:

$$p : M = \mathrm{Sh}_H \to \mathrm{Sh}_{\mathrm{GL}_2}, \; \iota : M = \mathrm{Sh}_H \to S = \mathrm{Sh}_G.$$

The construction of motivic classes

The construction II

$$\begin{array}{ccccccc}
\mathcal{B}_n & \xrightarrow{Eis^n_M} & \mathrm{H}^1_M(\mathrm{Sh}_{\mathrm{GL}_2}, \mathrm{Sym}^n V_2(1)) & \xrightarrow{p^*} & \mathrm{H}^1_M(M, W(1)) & \xrightarrow{\iota_*} & \mathrm{H}^3_M(S, V(2)) \\
\phi_f & \longmapsto & \mathrm{Eis}^n_M(\phi_f) & \longmapsto & p^*\mathrm{Eis}^n_M(\phi_f) & \longmapsto & c = \iota_* p^* \mathrm{Eis}^n_M(\phi_f)
\end{array}$$

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\end{array}
$$

Remark

- The construction is due to [D. Loeffler-C. Skinner-S. Zerbes 2022].
- When $V = \mathbb{Q}$, [A. Pollack-S. Shah 2018] gave an essentially similar construction of motivic classes.

The Hodge result

Notations

- $\mathcal{E}is^n_M := \iota_* \circ p^* \circ Eis^n_M$
- $\mathcal{E}is^n_H := r_H(\mathcal{E}is^n_M)$
- $\mathrm{H}^2_{B,!}(S, V(2)) := \mathrm{Im}(\mathrm{H}^2_{B,c}(S, V(2)) \to \mathrm{H}^2_B(S, V(2)))$

Theorem (S. 2024)

For suitable non-trivial algebraic representations V of G, the map $\mathcal{E}is^n_H : \mathcal{B}_{n,\mathbb{R}} \to \mathrm{H}^3_H(S, V(2))$ factors through the inclusion

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbf{1}, \mathrm{H}^2_{B,!}(S, V(2))) \hookrightarrow \mathrm{H}^3_H(S, V(2)),$$

where $\mathrm{MHS}^+_{\mathbb{R}}$ is the abelian category of mixed $\mathbb{R}$-Hodge structures and $\mathbf{1}$ denotes trivial Hodge structure, i.e., the unit of $\mathrm{MHS}^+_{\mathbb{R}}$.

Remarks on Theorem

Theorem (S. 2024)

For suitable non-trivial algebraic representations V of G, the map $\mathcal{E}is_H^n : \mathcal{B}_{n,\mathbb{R}} \to \mathrm{H}_H^3(S, V(2))$ factors through the inclusion

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where $\mathrm{MHS}^+_{\mathbb{R}}$ is the abelian category of mixed $\mathbb{R}$-Hodge structures and $\mathbf{1}$ denotes trivial Hodge structure, i.e., the unit of $\mathrm{MHS}^+_{\mathbb{R}}$.

Remark

- The proof uses a lot of Hodge theoretical computations.
- The Hodge theoretical vanishing on the boundary result for Eisenstein classes is also obtained in [G. Kings 1998] for Hilbert modular surfaces and in [F. Lemma 2015] for Siegel 3-folds. Our method is similar to theirs.

The motivic result

Theorem (S. 2024)

For suitable non-trivial alg. representations V of G, the motivic map $\mathcal{E}is^n_M : \mathcal{B}_n \to \mathrm{H}^3_M(S, V(2))$ factors through the inclusion

$$\mathrm{H}_M^{3+a-b+3(r-s)}(\mathrm{Gr}_0\mathrm{M}_{\mathrm{gm}}(S, V), \mathbb{Q}(2+a+2r-s)) \hookrightarrow \mathrm{H}^3_M(S, V(2)).$$

Remark

- $\mathrm{H}_M^{3+a-b+3(r-s)}(\mathrm{Gr}_0\mathrm{M}_{\mathrm{gm}}(S, V), \mathbb{Q}(2+a+2r-s))$ is the motivic incarnation for $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbf{1}, \mathrm{H}^2_{B,!}(S, V(2)))$, where a, b, r, s are the integer parameters defining V.
- G. Kings asked in 1998 whether we can prove the vanishing on the boundary for Eisenstein classes in the motivic world.
- My result is the first about vanishing on the boundary for Eisenstein classes in the motivic world.

Thank you!