Regularized periods of some Eisenstein series

Linli Shi

University of Connecticut

Purdue University
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Outline

1. Introduction
   - Modular forms
   - Divergence of period integrals
   - Regularized period integrals by Zagier
2. Toric periods
   - Automorphic forms
   - Toric periods of discrete spectrum
   - Toric periods of Eisenstein series
   - Regularized toric periods of Eisenstein series
3. Linear periods

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Modular curve

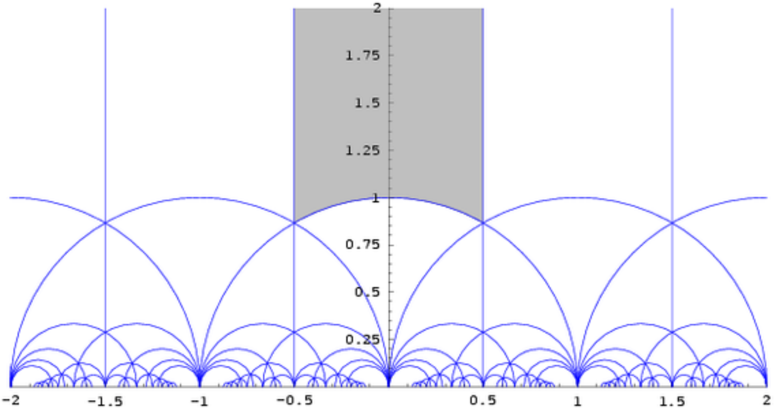
Definition

- Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} | y > 0\}$ be the upper half plane.
- Let Γ be the group $\mathrm{SL}_2(\mathbb{Z})$, each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ acts on $\mathcal{H}$ by linear fractional transformations:

$$z \mapsto \frac{az + b}{cz + d}.$$

- The quotient $\Gamma \backslash \mathcal{H}$ is a smooth manifold with real dimension 2 and it has a natural Γ-invariant measure $d\mu = dx \wedge dy / y^2$.
- $\Gamma \backslash \mathcal{H}$ is not compact, but $\mu(\Gamma \backslash \mathcal{H}) = \frac{\pi}{3} < \infty$.

Picture of the modular curve



Definition of modular forms

Definition

A modular form of weight $k \in \mathbb{Z}_{\geq 0}$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that:

- (Automorphy condition): for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have

$$f(\gamma(z)) = (cz+d)^k f(z), \text{ for all } \boxed{z = x + iy} \in \mathcal{H}.$$

- (Growth condition): $f(z)$ is bounded when $y \to \infty$.

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Remark

- $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma = \mathrm{SL}_2(\mathbb{Z}) \Rightarrow f(z+1) = f(z) \Rightarrow f(z) = \sum_{n \geq 0} a_n q^n$, where $q = e^{2\pi i z}$ and call $a_n \in \mathbb{C}$ the Fourier coefficients of f.
- A modular form where $a_0 = 0$ is called a cusp form.

Fourier coefficients of modular forms

Remark

- Here, $a_n e^{-2\pi ny}$ can be written as an integral:

$$f(q) = \sum_{n \geq 0} a_n q^n \Rightarrow \int_0^1 f(x+iy) e^{-2\pi inx} dx = a_n e^{-2\pi ny}.$$

- Hence, f is a cusp form if, taking $n = 0$,

$$\int_{\mathbb{Z}\backslash\mathbb{R}} f(x+iy) dx = \int_0^1 f(x+iy) dx = a_0 = 0.$$

- Later, we will generalize this definition of cusp forms to automorphic forms.

Examples of modular forms

Examples (Holomorphic Eisenstein series)

For even $k \geq 4$, the Eisenstein series $E_k(z)$ is defined as the absolutely convergent sum

$$E_k(z) = \frac{1}{2} \cdot \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k}.$$

It is a modular form of weight k and it has a Fourier expansion

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where $B_k \in \mathbb{Q}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1} \in \mathbb{Z}$. Since $a_0 = 1$, $E_k(z)$ is not a cusp form. But $E_4^3(z) - E_6^2(z) = 1728q + \cdots$ is a cusp form.

Non-holomorphic Eisenstein series

Generalization(non-holomorphic Eisenstein series)

- We can generalize Eisenstein series to the non-holomorphic setting, which can be seen as a "generalized modular form".
- For $z = x + iy \in \mathcal{H}$ and $s \in \mathbb{C}$, the non-holomorphic Eisenstein series $E(z,s)$ is defined for $\mathrm{Re}(s) > 1$ as :

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma(z))^s = y^s \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{|cz+d|^{2s}},$$

where $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Z} \right\}$ is the stabilizer of $i\infty$ in Γ. $E(z,s)$ is Γ-invariant: $E(\gamma(z),s) = E(z,s)$.The series is absolutely convergent for $\mathrm{Re}(s) > 1$ and $E(z,s)$ has a meromorphic continuation in s to $\mathbb{C}$ (z is fixed).

A formal computation

- A period integral is an integral of a function on a manifold over a closed submanifold. It has many applications in number theory.
- Ex: The integral $\int_0^\infty (\sum_{n=1}^\infty e^{-n^2\pi x}) x^{\frac{s}{2}-1} dx = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$ gives an integral representation of the Riemann ζ-function.
- Formally (without considering convergence), we have the following unfolding computation (recall $d\mu = \frac{dx \wedge dy}{y^2}$):

$$\int_{\Gamma \backslash \mathcal{H}} E(z,s) d\mu = \int_{\Gamma \backslash \mathcal{H}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma(z))^s d\mu = \int_{\Gamma_\infty \backslash \mathcal{H}} \mathrm{Im}(z)^s d\mu$$

$$= \int_0^\infty y^s \left(\int_{\mathbb{Z} \backslash \mathbb{R}} dx \right) \frac{dy}{y^2}$$

$$= \int_0^\infty y^{s-2} dy. \quad (\mathrm{Re}(s) > 1)$$

Divergence issue

Caution!

- We saw $\int_{\Gamma\backslash\mathcal{H}} E(z,s)d\mu = \int_0^\infty y^{s-2}dy.$ $(\mathrm{Re}(s) > 1)$.
- When $\mathrm{Re}(s) > 2$, $|y^{s-2}| \to \infty$ when $y \to \infty$.
- It is due to $E(z,s)$ is NOT rapidly decaying when $y \to \infty$.
- If $f(z)$ is a cusp form, $f(x+iy)$ rapidly decays as $y \to \infty$
- The integral
$$\int_0^\infty y^{s-2}dy \ (\mathrm{Re}(s) > 1)$$
diverges, so the above unfolding calculation is NOT rigorous.
- How to fix it?

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Regularized integral!

Regularizing the harmonic series

The harmonic series $\sum_{n\geq 1} 1/n$ diverges. Here are two ways to regularize it that both suggest assigning it the finite value $\gamma = .5772\ldots$, which is Euler's constant.

1. Truncate the series and subtract the divergent part. When N is large,

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + \frac{1}{2N} - \frac{1}{12N^2} + \cdots ,$$

so if we remove the divergent main term $\log N$ and then let $N \to \infty$, the right side tends to its constant term γ.

2. Insert a parameter and subtract the divergent part. When $s > 1$, the series $\sum_{n\geq 1} 1/n^s$ converges. When $s \to 1^+$,

$$\sum_{n\geq 1} \frac{1}{n^s} = \zeta(s) = \frac{1}{s-1} + \gamma + c_1(s-1) + c_2(s-1)^2 + \cdots ,$$

so if we remove the divergent main term $1/(s-1)$ and then let $s \to 1^+$, the right side tends to its constant term γ.

Fourier expansion of $E(z,s)$

- The intuition is to throw out some "unimportant" part of $E(z,s)$ for our purpose that leads to the divergence.
- Since $E(z,s)$ is Γ-invariant in z and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, with effect $z \mapsto z+1$, $E(z+1,s) = E(z,s)$. So $E(z,s)$ also has a Fourier expansion

$$E(z,s) = \sum_{n \geq 0} a_n(y,s) e^{2\pi i n x}. \ (z = x+iy)$$

- $a_n(y,s) = \int_0^1 E(z,s) e^{-2\pi i n x} dx$. By a computation, the constant term $a_0(y,s)$ is

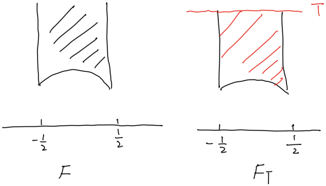
$$a_0(y,s) = y^s + \frac{\zeta^*(2s-1)}{\zeta^*(2s)} y^{1-s} \ (\mathrm{Re}(s) > 1),$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.
- $a_n(y,s)$ with $n > 0$ is rapidly decaying when $y \to \infty$.

Truncation operator I

Recall $a_0(y,s) = y^s + \frac{\zeta^*(2s-1)}{\zeta^*(2s)} y^{1-s}$ $(\mathrm{Re}(s) > 1)$.

- Since $a_0(y,s)$ grows like $|y^s|$ when $\mathrm{Re}(s) > 1$, which is not rapidly decaying, a natural idea is to throw out $a_0(y,s)$ and integrate the remaining terms.
- This leads to the definition of a truncation operator.
- Let $\mathcal{F} = \{z \in \mathcal{H} \,|\, |z| \geq 1, |x| \leq \frac{1}{2}\}$ be a fundamental domain of $\Gamma \backslash \mathcal{H}$.
- For $T \geq 1$, let $\mathcal{F}_T = \{z \in \mathcal{H} \,|\, |z| \geq 1, |x| \leq \frac{1}{2}, y \leq T\}$ be a truncated fundamental domain.



Truncation operator II

Definition (Zagier 1981)

- Let χ_T be the indicator function of $\mathcal{F}_T$: for $z \in \mathcal{F}$,

$$\chi_T(z) = \begin{cases} 1 & \text{if } z \in \mathcal{F}_T, \\ 0 & \text{if } z \notin \mathcal{F}_T. \end{cases}$$

- For $T \geq 1$ and $F(z) = \sum_{n \geq 0} a_n(y) e^{2\pi i x}$ smooth on $\mathcal{F}$, the truncation operator Λ^T on F is defined as

$$(\Lambda^T F)(z) = F(z) - (1 - \chi_T(z)) a_0(y), \text{ for } z = x + iy.$$

- $(\Lambda^T F)(z)$ is a function that decays rapidly as $y \to \infty$ (x fixed).

Regularized integrals

Theorem (Zagier 1981)

- *The integral* $\int_{\mathcal{F}} \Lambda^T(E(z,s))d\mu(z)$ *abs. conv. and equals*

$$P(T) := \frac{1}{s-1}T^{s-1} - \frac{\zeta^*(2s-1)}{s\cdot\zeta^*(2s)}T^{-s}.$$

- *The regularized integral* $\int_{\mathcal{F}}^* E(z,s)d\mu$ *is defined as the constant term of* $P(T)$.
- *This suggests*

$$\int_{\mathcal{F}}^* E(z,s)d\mu = 0.$$

Remark

The regularized period integral throws out $\int_0^\infty y^{s-2}dy$ in the formal computation.

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p-adic fields

- There are other kinds of absolute values on $\mathbb{Q}$: the p-adic absolute values $|\cdot|_p$ for prime p. Denote the usual absolute value as $|\cdot|_\infty$.
- Let $a = p^m b \in \mathbb{Z} - \{0\}$, where $p, b, m \in \mathbb{Z}$ and $(p, b) = 1$. The p-adic absolute value of a is defined as $|a|_p = \frac{1}{p^m}$ (e.g. $|18|_\infty = 18$, $|18|_3 = \frac{1}{9}$, $|18|_5 = 1$.)
- The completion of $\mathbb{Q}$ wrt $|\cdot|_p$ is the p-adic field $\mathbb{Q}_p$.
- $\mathbb{Q}_p$ is a locally compact topological field.
- The ring of p-adic integers $\mathbb{Z}_p$ is defined as

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p | |x|_p \leq 1\}.$$

- $\mathbb{Z}_p$ is a compact topological ring.
- **Fact** (Ostrowski 1916) Every nontrivial absolute value on $\mathbb{Q}$ is equivalent (induces the same topology) to some $|\cdot|_p$ or $|\cdot|_\infty$.

Ring of Adeles

- We want a "machine" that contains $\mathbb{Q}_p$ for all prime p and $\mathbb{R}$.
- The natural guess is $\mathbb{R} \times \prod_p \mathbb{Q}_p$. However, this is too large: it is not locally compact for the product topology.
- Restrict coordinates to get the adeles $\mathbb{A}_\mathbb{Q}$:

$$\mathbb{A}_\mathbb{Q} := \left\{ (x_\infty, x_2, x_3, x_5, \cdots) \in \mathbb{R} \times \prod_p \mathbb{Q}_p \middle| x_p \in \mathbb{Z}_p, \text{ for a.e. } p \right\},$$

  here *a.e.* means almost every prime: all but finitely many p.
- $\mathbb{A}_\mathbb{Q}$ is a locally compact ring in a suitable topology.
- $\mathbb{Z} \subset \mathbb{R}$ discrete, $\mathbb{Z}\backslash\mathbb{R}$ compact; $\mathbb{Q} \subset \mathbb{A}_\mathbb{Q}$ discrete, $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ compact.
- $\mathbb{Q}^\times \subset \mathbb{A}_\mathbb{Q}^\times$ discrete, $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times$ NOT compact.
- $(\mathbb{A}_\mathbb{Q}^\times)^1 := \{x \in \mathbb{A}_\mathbb{Q}^\times \,|\, |x| = 1\}$, $\mathbb{Q}\backslash(\mathbb{A}_\mathbb{Q}^\times)^1$ compact.

Automorphic forms on GL_2, I

- Let $B_2 = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2 \right\}$, $T_2 = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2 \right\}$, $N_2 = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2 \right\}$. We call T_2 a torus.
- Let $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})^1 := \{g \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) || \det(g)| = 1\}$, which is a locally compact topological group.
- $\mathrm{GL}_2(\mathbb{Q})$ is discrete in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})^1$, like $\mathbb{Q}^\times$ in $(\mathbb{A}_{\mathbb{Q}}^\times)^1$.
- The quotient space $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})^1$ has a natural invariant measure dg and has finite volume like $\Gamma\backslash\mathcal{H}$. But it is not compact.
- An *automorphic form* is a "nice" smooth $\mathbb{C}$-valued function on $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})^1$. This generalizes modular forms.
- A *cusp form* ϕ is an automorphic form that satisfies

$$\int_{N_2(\mathbb{Q})\backslash N_2(\mathbb{A}_{\mathbb{Q}})} \phi(ng)dn = 0.$$

This generalizes cuspidal modular forms.

Automorphic forms on GL_2, II

- Let $[\mathrm{GL}_2]$ denote $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})^1$.
- Since $[\mathrm{GL}_2]$ is not compact, get

$$\mathcal{A}([\mathrm{GL}_2])) = \mathcal{A}_{disc}([\mathrm{GL}_2]) \oplus \mathcal{A}_{cont}([\mathrm{GL}_2]).$$

- The cusp forms span a closed subspace $\mathcal{A}_{\mathrm{cusp}}([\mathrm{GL}_2])$ of $\mathcal{A}([\mathrm{GL}_2])$.
- **Fact (Gelfand and Piatetski-Shapiro)**

$$\mathcal{A}_{\mathrm{cusp}}([\mathrm{GL}_2]) \subsetneq \mathcal{A}_{disc}([\mathrm{GL}_2]).$$

- **Fact (Moeglin and Waldspurger)** The orthogonal complement of $\mathcal{A}_{\mathrm{cusp}}([\mathrm{GL}_2])$ in $\mathcal{A}_{disc}([\mathrm{GL}_2])$ can be constructed using residues of some Eisenstein series.
- **Fact (Langlands)**:
  $\mathcal{A}_{cont}([\mathrm{GL}_2])$ is "spanned by" Eisenstein series.

Automorphic forms on GL_2, III

Definition

- Let $A_{B_2}^\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) | a > 0, d > 0 \right\}$ and let $\mathfrak{a}_{B_2} = \mathrm{Lie}(A_{B_2}^\infty) \cong \mathbb{R}^2$, $\mathfrak{a}_{B_2}^0 \cong \mathfrak{a}_{B_2}/\mathfrak{a}_0 \cong \mathbb{R}$.
- There is a canonical map $H_{B_2} : \mathrm{GL}_2(\mathbb{A})^1 \to \mathfrak{a}_{B_2}^0$.
- For each $\varphi \in \mathcal{A}_{\mathrm{cusp}}(A_{B_2}^\infty T_2(\mathbb{Q}) N_2(\mathbb{A}_\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})^1)$, $g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})^1$ and $\lambda \in \mathfrak{a}_{B_2,\mathbb{C}}^{0,*}$, the Eisenstein series $E(g, \varphi, \lambda)$ is defined as

$$E(g, \varphi, \lambda) = \sum_{\gamma \in B_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{Q})} \varphi(\gamma g) \exp \langle \lambda, H_{B_2}(\gamma g) \rangle.$$

- The summation in the definition is absolutely convergent if $\mathrm{Re}(\lambda)$ is sufficiently large.
- $E(g, \varphi, \lambda)$ can be meromorphic continued to $\mathbb{C}$.

Toric periods of discrete spectrum

- Let $G = \mathrm{GL}_2$ and $H = T_2 = \{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)\} \subset \mathrm{GL}_2$ (H is a torus).
- For an automorphic form ϕ on G, we are interested in the toric period:

$$P_H(\phi) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})\cap G(\mathbb{A}_\mathbb{Q})^1} \phi(h)dh.$$

- If ϕ is a cusp form, this is absolutely convergent [Ash, Ginzburg, Rallis 1993]. In 1993, Friedberg and Jacquet proved a relation between $P_H(\phi)$ and special values of some L-function.
- If ϕ is a noncuspidal automorphic form in the discrete spectrum, Yang computed the regularized toric period and proved that it is factorizable in 2022.
- Hence, only $P_H(\phi)$ for ϕ in the continuous spectrum remains to be considered.

Formal computations

- We begin by formal unfolding computations.
- Let $\varphi_\lambda(g)=\varphi(g)\exp(\langle\lambda, H_{B_2}(g)\rangle)$ and $H_\eta = H\cap\eta^{-1}B_2\eta$ for $\eta\in G(\mathbb{Q})$.
- $$\begin{aligned}
&\int_{H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})\cap G(\mathbb{A}_\mathbb{Q})^1} E(g,\varphi,\lambda)dh \\
&=\int_{H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})\cap G(\mathbb{A}_\mathbb{Q})^1} \sum_{\gamma\in B_2(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_\lambda(\gamma h)dh \\
&=\int_{H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})\cap G(\mathbb{A}_\mathbb{Q})^1} \sum_{\eta\in B_2(\mathbb{Q})\backslash G(\mathbb{Q})/H(\mathbb{Q})} \sum_{\gamma\in H_\eta(\mathbb{Q})\backslash H(\mathbb{Q})} \varphi_\lambda(\eta\gamma h)dh \\
&=\sum_{\eta\in B_2(\mathbb{Q})\backslash G(\mathbb{Q})/H(\mathbb{Q})} \int_{H_\eta(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})\cap G(\mathbb{A}_\mathbb{Q})^1} \varphi_\lambda(\eta h)dh.
\end{aligned}$$

Another type of period integrals

Definition

For $\eta \in B_2(\mathbb{Q})\backslash G(\mathbb{Q})/H(\mathbb{Q})$, $\varphi \in \mathcal{A}_{\mathrm{cusp}}(A_{B_2}^{\infty} T_2(\mathbb{Q}) N_2(\mathbb{A}_{\mathbb{Q}})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})^1)$ and $\lambda \in \mathfrak{a}_{B_2,\mathbb{C}}^{0,*}$, the integral

$$J(\eta, \varphi, \lambda) := \int_{H_\eta(\mathbb{Q})\backslash H(\mathbb{A}_{\mathbb{Q}})\cap G(\mathbb{A}_{\mathbb{Q}})^1} \varphi_\lambda(\eta h) dh.$$

is called the *intertwining period* associated to η.

Theorem (S. 2024)

$J(\eta, \varphi, \lambda)$ is absolutely convergent for $\mathrm{Re}(\lambda)$ is sufficiently large.

Regularized periods, I

- In 2019, Zydor defined a **relative** truncation operator Λ^T for T in suitable open cone of $\mathfrak{a}_{B_2}^0$ as an operator from the space of moderate growth functions on $G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})^1$ to the space of rapidly decaying functions on $H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathbb{Q}}) \cap G(\mathbb{A}_{\mathbb{Q}})^1$.
- It is a vast generalization of the truncation used by Zagier.
- For any automorphic form ϕ on G, Zydor showed the integral

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathbb{Q}})\cap G(\mathbb{A}_{\mathbb{Q}})^1} (\Lambda^T \phi)(h)dh$$

is absolutely convergent.

Regularized periods, II

Theorem (Zydor 2019)

- *Fix some "regular" ϕ. The truncated period integral $\int_{H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})\cap G(\mathbb{A}_\mathbb{Q})^1}(\Lambda^T\phi)(h)dh$ is an exponential polynomial function of T. Explicitly, it equals a finite sum*

$$\sum_{\lambda\in\mathfrak{a}^{0,*}_{B_2,\mathbb{C}}} P_\lambda(T)\exp(\langle\lambda, T\rangle),$$

  where $P_\lambda(T)$ is a polynomial in T.
- *$P_0(T)$ is a constant.*
- *The regularized period integral is defined as*

$$\int^*_{H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})\cap G(\mathbb{A}_\mathbb{Q})^1}\phi(h)dh := P_0(T).$$

The main result for GL_2

Theorem (S. 2024)

We have the following identity:

$$\int^{*}_{H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathbb{Q}})\cap G(\mathbb{A}_{\mathbb{Q}})^1} E(h,\varphi,\lambda)dh = J(\eta,\varphi,\lambda),$$

for some $\eta \in B_2(\mathbb{Q})\backslash G(\mathbb{Q})/H(\mathbb{Q})$ in the open orbit.

Remark

- The theorem gives a rigorous justification of the "unfolding".
- By this identity and meromorphic continuation and functional equation of $E(h,\varphi,\lambda)$, we can get the meromorphic continuation and functional equation of $J(\eta,\varphi,\lambda)$.
- The work for GL_2 and $T_2 = \mathrm{GL}_1 \times \mathrm{GL}_1$ extends to GL_{n+m} and $\mathrm{GL}_n \times \mathrm{GL}_m$, which will be stated.

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The regularized periods for cuspidal Eisenstein series

Theorem (S. 2024)

1. *Let* $G = \mathrm{GL}_{2n}$ *and* $H = \mathrm{GL}_n \times \mathrm{GL}_n$ *for* $n \in \mathbb{Z}_{\geq 1}$. *For any standard parabolic subgroup* $P = MN$ *of* G, *any* $\varphi \in \mathcal{A}_{P,cusp}(G)$ *and* $\lambda \in \mathfrak{a}^{0,*}_{M,\mathbb{C}}$, *we have*

$$\int^{*}_{H(F)\backslash H(\mathbb{A}_F)^{G,1}} E(h,\varphi,\lambda)dh = \begin{cases} 0 & M \neq \mathrm{GL}_n \times \mathrm{GL}_n, \\ J(\eta,\varphi,\lambda) & M = \mathrm{GL}_n \times \mathrm{GL}_n, \end{cases}$$

where η *the open orbit in* $P(F)\backslash G(F)/H(F)$.

2. *Let* $G = \mathrm{GL}_{n+m}$ *and* $H = \mathrm{GL}_n \times \mathrm{GL}_m$ *for* $n, m \in \mathbb{Z}_{\geq 1}$ *and* $n \neq m$. *For any standard parabolic subgroup* $P = MN$ *of* G, *any* $\varphi \in \mathcal{A}_{P,cusp}(G)$ *and* $\lambda \in \mathfrak{a}^{0,*}_{M,\mathbb{C}}$, *we have*

$$\int^{*}_{H(F)\backslash H(\mathbb{A}_F)^{G,1}} E(h,\varphi,\lambda)dh = 0.$$

The regularized periods for discrete Eisenstein series

Theorem (S. 2024)

Let $G = \mathrm{GL}_{2n}$ and $H = \mathrm{GL}_n \times \mathrm{GL}_n$ for $n \in \mathbb{Z}_{\geq 1}$. For any standard parabolic subgroup $P = MN$ of G, any $\varphi \in \mathcal{A}_{P,\pi}(G)$ with $\pi \in \Pi_{disc}(M)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^{0,*}$, we have:
The regularized periods

$$\int^*_{H(F)\backslash H(\mathbb{A}_F)^{G,1}} E(h, \varphi, \lambda) dh = 0.$$

except the following two cases.

1. *When $M = G$ and $\pi \cong \pi^\vee$. (Computation by C. Yang.)*
2. *When $M = \mathrm{GL}_n \times \mathrm{GL}_n$ and $\pi \cong \sigma \boxtimes \sigma^\vee$, where $\sigma \in \Pi_{disc}(\mathrm{GL}_n)$. In this case, we have*

$$\int^*_{H(F)\backslash H(\mathbb{A}_F)^{G,1}} E(h, \varphi, \lambda) dh = J(\eta, \varphi, \lambda).$$

Remarks I

Remark

- By multiplicity one, for $\varphi = \otimes'_v \varphi_v$, the global intertwining periods can be factorized into products of local intertwining periods:
$$J(\eta, \varphi, \lambda) = \prod_v J_v(\eta, \varphi_v, \lambda).$$
- (Suzuki-Xue, Offen, Lapid-Offen) For v unramified, we have
$$J_v(\eta, \varphi_v, \lambda) = \frac{L(2\lambda, \pi_v, \wedge^2) L(\lambda + \frac{1}{2}, \pi_v) L(\lambda + \frac{1}{2}, \pi_v \otimes \eta_v)}{L(2\lambda + 1, \pi_v, \mathrm{Sym}^2)}.$$
- The absolutely convergence, functional equations and meromorphic continuation of local intertwining periods was proved by Matringe-Offen-Yang.
- From above, we can give sufficient and necessary conditions for regularized linear periods.

Final remarks

Remark

- In 2022, Suzuki and Xue computed the regularized periods of cuspidal Eisenstein series for $D_E \backslash \mathrm{GL}_2(D)$, where D is a division algebra over F and E is a quadratic extension of F. Hence, our result for cuspidal Eisenstein series can be viewed as a split counterpart of the result of Suzuki and Xue.
- We have a work in progress to get a **global** spectral expansion for $\mathrm{GL}_n \times \mathrm{GL}_n \backslash \mathrm{GL}_{2n}$ using our results.
- The author has another work in progress with Yiyang Wang (Kyoto) to find a new regularization for $\mathrm{GL}_n \times \mathrm{GL}_m \backslash \mathrm{GL}_{n+m}$, which has applications for proving a **local** Plancherel formula for $\mathrm{GL}_{n+m} / \mathrm{GL}_n \times \mathrm{GL}_m$.

Thank you!